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Abstract. We consider a damped impact oscillator subject to the action of a biharmonic force. The conditions for the existence and stability of almost periodic resonance solutions are investigated.

1 Introduction

Representations of vibro-impact processes in terms of integrals of motion for vibro-impact system with one degree of freedom is determined by the physics of vibro-impact system. The corresponding variables (impulse - phase) are very natural for the vibrating processes accompanied by impacts. The impulse can be interpreted as the force characteristics of impact. The phase is the instant of collision.

In case of an instantaneous impact the representations of periodic vibro-impact processes can be written via so-called periodic Green's functions. Periodic Green's functions are determined only by the linear part of the system. They appear below or in the form of Fourier series or finite relations on intervals of periodicity. The impact force is represented by the Dirac δ -function and the values depend on the impulse and phase of the impact. The periodic Green's functions are calculated using well known methods [2]. Similar approach was proposed by V.I. Babitsky and M.Z. Kolovsky [3].

We want to give a correct description of resonant modes in vibro-impact system with one degree of freedom.

We chose to study a model with elastic impact and viscous damping. Such a model is discussed in [2], [14].

A model that takes into account the energy loss at impact is proposed in [2]. It is not clear how to explore this model on a mathematical level of rigor.

Our study is based on a combination of the method of averaging on infinite interval [6] and the methods of singular perturbation theory [10,13].

We will study the motion of conservative vibro-impact systems with one degree of freedom and subject to small biharmonic perturbations. This problem is reduced to study two-dimensional systems with fast rotating phase and slow varying coefficients.

Periodic perturbations of smooth two-dimensional systems with fast rotating phase and slow varying coefficients were studied in [9]. The conditions of closeness of solutions of exact and averaged equations on a finite asymptotically large time interval are established. Resonance almost periodic oscillations in such systems are investigated in [5] and [6].

Recently, much attention has been focused on studies of dynamics of nonlinear systems perturbed by a biharmonic external force with different frequencies (see, for example [1, 4, 7, 8]).

2 Conservative impact oscillator

Here we follow the method from the book [2]. Consider a linear oscillator

$$\ddot{x} + \Omega^2 x = 0.$$

At the point $x = \Delta$, we arrange an immovable limiter and assume that once the coordinate x reaches the value Δ , an instant elastic impact occurs in the system so that if $x = \Delta$ at the time instant t_α , then the relation

$$\dot{x}(t_\alpha - 0) = -\dot{x}(t_\alpha + 0) \quad (1)$$

holds.

If $\Delta > 0$ and the energy level in the linear system is insufficient to attain the level $x = \Delta$, then linear oscillations with the frequency Ω take place. In the presence of collisions, the oscillation frequency satisfies the inequality $\omega > \Omega$ and rises as the energy rises but does not exceed the value 2Ω . Hence

$$\Omega < \omega < 2\Omega, \quad \Delta > 0. \quad (2)$$

If $\Delta < 0$, the oscillation frequency ω obeys the inequality

$$2\Omega < \omega < \infty, \quad \Delta < 0. \quad (3)$$

At $\Delta = 0$ the image point passes any phase trajectory for the same time with the doubled velocity 2Ω so that

$$\omega = 2\Omega, \quad \Delta = 0. \quad (4)$$

Condition (1) suggests that variation of the impulse Φ_0 in the neighborhood of the impact instant t_α takes the form

$$J = \dot{x}_- - \dot{x}_+ = 2\dot{x}_-, \quad \dot{x}_- > 0,$$

where $\dot{x}_\alpha = \dot{x}(t_\alpha \mp 0)$.

The resulting force becomes localized at $t = t_\alpha$. Hence

$$\Phi_0|_{t=t_\alpha} = J\delta(t - t_\alpha) \quad (5)$$

and

$$\int_{t_\alpha-0}^{t_\alpha+0} \Phi_0 dt = J.$$

Impacts occur periodically when $t_\alpha = t_0 + \alpha T$, where α is an integer, and T is the period between impacts calculated by the equality $T = 2\pi\omega^{-1}$ and (2)–(4). Thus, for $-\infty < t < \infty$, we obtain a T -periodic continuation of (5)

$$\Phi_0 = J\delta_T(t - t_0),$$

where $\delta_T(t)$ is the T -periodic δ -function.

Solution of the equation

$$\ddot{x} + \Omega^2 x + \Phi_0(x, \dot{x}) = 0 \quad (6)$$

is understood as the T -periodic function $x(t)$ such that its substitution into this equation transforms it into a correct equality (from the viewpoint of the theory of distributions) of the form

$$\ddot{x} + \Omega^2 x + J\delta_T(t - t_0) = 0,$$

where t_0 is an arbitrary constant, and for all $\alpha = 0, \pm 1, \dots$

$$x(t_0 + \alpha T) = \Delta, \quad J = 2\dot{x}_-(t_0 + \alpha T).$$

At the same time, the restrictions

$$x(t) \leq \Delta, \quad \dot{x}_- > 0$$

are fulfilled, and the periods of oscillations, depending on the sign of Δ , fit the frequency ranges of (2)–(4).

To describe the solution analytically, we assume $t_0 = 0$. In this case, for $0 \leq t < T_0$, the solution of equation (6) has the form

$$x(t) = -J\kappa[\omega_0(J)(t - t_0), \omega_0(J)], \quad \kappa(t, \omega_0) = \frac{1}{2\Omega} \frac{\cos[\Omega(t - T_0/2)]}{\sin(\Omega T_0/2)} = \\ \frac{\omega_0}{2\pi\Omega^2} + \frac{\omega_0}{\pi} \sum_{k=1}^{\infty} \frac{\cos k\omega_0 t}{\Omega^2 - k^2\omega_0^2}, \quad J(\omega_0) = -2\Omega\Delta \tan \frac{\Omega T_0}{2}, \quad J \geq 0,$$

and the third relation here determines the smooth dependence $\omega_0(J)$ at $\Delta \neq 0$, whereas $\omega_0 = 2\Omega$ at $\Delta = 0$.

Geometric conditions of an impact result in frequency intervals (2)–(4). Note that when $\Delta = 0$, the solution $x(t)$ for $0 \leq t < \pi/\Omega$ takes the form

$$x(t) = -\frac{J}{2\Omega} \sin \Omega t,$$

where J is a frequency-independent arbitrary constant.

3 Perturbed impact oscillator

Now consider a perturbed impact oscillator

$$\ddot{x} + \Omega^2 x + \Phi_0(x, \dot{x}) = \varepsilon[f(t, \tau) - \gamma\dot{x}], \quad (7)$$

where $\varepsilon > 0$ is a small parameter, $\tau = \varepsilon t$ is a slow time, $\gamma > 0$ is a constant. The function $\Phi_0(x, \dot{x})$ describes the force of impact. Representation of this function is given in the previous section.

We consider the perturbations of two types. First a biharmonic perturbation is the sum of two small periodic forces with close frequencies. The corresponding perturbation has the form

$$f(t, \tau) = a_1 \sin \nu t + a_2 \sin(\nu t + \Gamma\tau),$$

where a_1, a_2, ν, Γ are real positive numbers. The perturbation function $f(t, \tau)$ can be represented as

$$f(t, \tau) = E(\tau) \sin(\nu t + \beta(\tau)), \quad \tau = \varepsilon t, \quad (8)$$

where

$$E(\tau) = \sqrt{a_1^2 + 2a_1a_2 \cos \Gamma\tau + a_2^2}, \quad \cos \beta(\tau) = \frac{a_1 + a_2 \cos \Gamma\tau}{E(\tau)}, \\ \sin \beta(\tau) = \frac{a_2 \sin \Gamma\tau}{E(\tau)}.$$

The function (8) is periodic in t with the period $2\pi/\nu$ and periodic in τ with the period $2\pi/\Gamma$. The function $E(\tau)$ is strictly positive if $a_1 \neq a_2$, which will be assumed. Consequently the function $f(t, \varepsilon t)$ is almost periodic function (see [6]) with two basic frequencies.

Second perturbation is a biharmonic force with very different frequencies. This force has the form

$$f(t, \tau) = A \sin(\nu t + \theta) + B \sin \Gamma\tau, \quad \tau = \varepsilon t. \quad (9)$$

where $A, B, \nu, \Gamma, \theta$ are real numbers.

We assume $\psi = \omega_0(J)t$ and transform equation (7) into the system in terms of the variables J, ψ (impulse-phase), by making a change

$$x = -J\kappa[\psi, \omega_0(J)], \\ \dot{x} = -J\omega_0(J)\kappa_\psi[\psi, \omega_0(J)], \quad (10)$$

where

$$\kappa[\psi, \omega_0(J)] = \kappa(\psi, J) = \omega_0(J)^{-1} \left[\frac{1}{2\pi\Omega_0^2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\cos k\psi}{\Omega_0^2 - k^2} \right], \quad \Omega_0 = \frac{\Omega}{\omega_0(J)}.$$

It follows from the theory of Fourier series (see, for example, [12, chapter 4]) that the function $\kappa_\psi[\psi, \omega_0(J)]$ has a finite discontinuities at the points $\psi = 2l\pi$, where l is an integer, and continuous at all other points. Moreover the function $\kappa_\psi[\psi, \omega_0(J)]$ has a derivative at interior points of the intervals $[2\pi l, 2\pi(l+1)]$. Fourier series of function $\kappa_\psi[\psi, \omega_0(J)]$ has a form

$$-\omega_0(J_{pq})^{-1} \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{k \sin k\psi}{\Omega_0^2 - k^2}.$$

Therefore change (10) is not smooth at $\psi = 2l\pi$. Thus, in new variables, impacts occur when $\psi = 2l\pi$. Making the substitution (10), we arrive at the system

$$\frac{dJ}{dt} = -4\varepsilon\omega_0(J)[f(t, \tau) + \gamma J\omega_0(J)\kappa_\psi(\psi, J)]\kappa_\psi(\psi, J), \\ \frac{d\psi}{dt} = \omega_0(J) - 4\varepsilon\omega_0(J)J^{-1}[f(t, \tau) + J\omega_0(J)\kappa_\psi(\psi, J)](-J\kappa(\psi, J))_J. \quad (11)$$

A detailed derivation of the system (11) is contained in the book [2, chapter 2]. This is a system with a fast rotating phase, where the right-hand sides are periodic in ψ and have finite discontinuities at the points $\psi = 2l\pi$. The dependencies $\omega_0(J)$ have the form

$$\omega_0(J) = \frac{\pi\Omega}{\pi - \arctan[J/(2\Omega\Delta)]}, \quad \Delta > 0, \quad \Omega < \omega_0 < 2\Omega, \\ \omega_0(J) = -\frac{\pi\Omega}{\arctan[J/(2\Omega\Delta)]}, \quad \Delta < 0, \quad 2\Omega < \omega_0 < \infty, \\ \omega_0 = 2\Omega = \text{const}, \quad \Delta = 0.$$

System (11) is a system with two slow variables J, τ and two fast variables ψ, t . The existence and stability of stationary resonance solutions in such systems will be investigated.

4 Construction of averaged equations

We will use the method of averaging on an infinite interval (see [6]). The major part in all the problems related to the principle of averaging is in the changes of variables. The changes allow to eliminate fast variables from equations of motion within the given accuracy and thus separate slow motion from the fast one.

Let J_{pq} be a solution of the equation

$$\omega_0(J_{pq}) = \frac{q}{p}\nu.$$

where p, q are relatively prime integers. By making a change $\psi = \varphi + \frac{q}{p}\nu t$ we transform system (11) into

$$\begin{aligned} \frac{dJ}{dt} &= -4\varepsilon\omega_0(J)[f(t, \tau) + \gamma J\omega_0(J)\kappa_\psi(\varphi + \frac{q}{p}\nu t, J)]\kappa_\psi(\varphi + \frac{q}{p}\nu t, J), \\ \frac{d\varphi}{dt} &= \omega_0(J) - \frac{q}{p}\nu - 4\varepsilon\omega_0(J)J^{-1}[f(t, \tau) + \gamma J\omega_0(J)\kappa_\psi(\varphi + \frac{q}{p}\nu t, J)] \times \\ &\quad (-J\kappa(\varphi + \frac{q}{p}\nu t, J))_J. \end{aligned} \quad (12)$$

The point $J = J_{pq}$ is resonant point in system (12).

Assume that the resonance is non-degenerate, i.e.,

$$\left. \frac{d\omega_0}{dJ} \right|_{J=J_{pq}} = \omega'_0(J_{pq}) \neq 0. \quad (13)$$

We shall study the behavior of solutions of system (12) in the $\mu = \sqrt{\varepsilon}$ -neighborhood of the resonance point J_{pq} . We make a change

$$J = J_{pq} + \mu z$$

and expand the right-hand side of system (12) in terms of the powers of μ . As a result, we obtain the system

$$\begin{aligned} \frac{dz}{dt} &= \mu F_0(t, \tau, \varphi, J_{pq}) + \mu^2 F_1(t, \tau, \varphi, J_{pq})z + O(\mu^3), \\ \frac{d\varphi}{dt} &= \mu\omega'_0(J_{pq})z + \frac{1}{2}\mu^2\omega''_0(J_{pq})z^2 + \mu^2 G_0(t, \tau, \varphi, J_{pq}) + O(\mu^3) \end{aligned} \quad (14)$$

where

$$F_0(t, \tau, \varphi, J_{pq}) = -4\omega_0(J_{pq})[f(t, \tau) + \gamma J_{pq}\omega_0(J_{pq})\kappa_\psi(\varphi + \frac{q}{p}\nu t, J + pq)] \times \kappa_\psi(\varphi + \frac{q}{p}\nu t, J_{pq}),$$

$$F_1(t, \tau, \varphi, J_{pq}) = -4\frac{d}{dJ} \left\{ \omega_0(J)[f(t, \tau) + \gamma J\omega_0(J)\kappa_\psi(\varphi + \frac{q}{p}\nu t, J)] \times \kappa_\psi(\varphi + \frac{q}{p}\nu t, J) \right\} \Big|_{J=J_{pq}}.$$

$$G_0(t, \tau, \varphi, J_{pq}) = -4\omega_0(J_{pq})J_{pq}^{-1}[f(t, \tau) + \gamma J_{pq}\omega_0(J_{pq})\kappa_\psi(\varphi + \frac{q}{p}\nu t, J_{pq})] \times (-\kappa(\varphi + \frac{q}{p}\nu t, J_{pq}) - J_{pq}\kappa_J(\varphi + \frac{q}{p}\nu t, J_{pq})).$$

System (14) contains only one fast variable t . We now make the standard change of the method of averaging in order to eliminate the fast variable in the right-hand side of system (14) up to the accuracy of the terms of order μ^2 . This change is sought in the form

$$z = \xi + \mu u_1(\eta, t, \tau) + \mu^2 u_2(\eta, t, \tau)\xi, \quad \varphi = \eta + \mu^2 v_2(\eta, t, \tau),$$

where the functions $u_i(\eta, t, \tau)$, ($i = 1, 2$), $v_2(\eta, t, \tau)$ are periodic in t, τ with period $2\pi/\nu$ and $2\pi/\Gamma$ accordingly.

The change results in the system

$$\begin{aligned} \frac{d\xi}{dt} &= \mu f_0(\eta, \tau) + \mu^2 f_1(\eta, \tau)\xi + O(\mu^3), \\ \frac{d\eta}{dt} &= \mu \omega'_0(J_{pq})\xi + \frac{1}{2}\mu^2 \omega''_0(J_{pq})\xi^2 + \mu^2 g_0(\eta, \tau) + O(\mu^3), \end{aligned} \quad (15)$$

where $f_0(\eta, \tau)$, $f_1(\eta, \tau)$, $g_0(\eta, \tau)$ are defined as the mean values over t :

$$\begin{aligned} f_0(\eta, \tau) &= \frac{\nu}{2\pi} \int_0^{2\pi/\nu} F_0(t, \tau, \eta, J_{pq}) dt, \\ f_1(\eta, \tau) &= \frac{\nu}{2\pi} \int_0^{2\pi/\nu} F_1(t, \tau, \eta, J_{pq}) dt, \\ g_0(\eta, \tau) &= \frac{\nu}{2\pi} \int_0^{2\pi/\nu} G_0(t, \tau, \eta, J_{pq}) dt. \end{aligned}$$

The functions $u_i(\eta, t, \tau)$, ($i = 1, 2$), $v_2(\eta, t, \tau)$ are defined as periodic solutions in t with the zero mean value from the equations

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= F_0(t, \tau, \eta, J_{pq}) - f_0(\eta, \tau), \\ \frac{\partial u_2}{\partial t} &= F_1(t, \tau, \eta, J_{pq}) - u_1(\eta, t, \tau)\omega_0(J_{pq}) - f_1(\eta, \tau), \\ \frac{\partial v_2}{\partial t} &= G_0(t, \tau, \eta, J_{pq}) - u_1(\eta, t, \tau)\omega_0(J_{pq}) - g_0(\eta, \tau). \end{aligned}$$

Functions $f_0(\eta, \tau)$, $f_1(\eta, \tau)$, $g_0(\eta, \tau)$ are smooth in η for considered perturbations. The function $f_0(\eta, \tau)$ is mean value over t of the function $F_0(t, \tau, \varphi, J_{pq})$. After averaging only a single term remains from the infinite sum. Calculation of $f_0(\eta, \tau)$ will be demonstrated below (see (34)). Similar assertions hold for functions $f_1(\eta, \tau)$, $g_0(\eta, \tau)$.

System (15) at the time τ is a singularly perturbed system in the following form

$$\begin{aligned} \mu \frac{d\xi}{d\tau} &= f_0(\eta, \tau) + \mu f_1(\eta, \tau)\xi + O(\mu^2), \\ \mu \frac{d\eta}{d\tau} &= \omega'_0(J_{pq})\xi + \frac{1}{2}\mu \omega''_0(J_{pq})\xi^2 + \mu g_0(\eta, \tau) + O(\mu^2), \end{aligned} \quad (16)$$

5 Existence and stability of almost periodic solutions

Let there exist periodic function $\eta_0(\tau)$ such that

$$f_0(\eta_0(\tau), \tau) \equiv 0, \quad 0 < \eta_0(\tau) < 2\pi. \quad (17)$$

In this case, the degenerate system derived from (16) at $\mu = 0$ has the solution

$$\xi = 0, \quad \eta = \eta_0(\tau). \quad (18)$$

Linearizing the right-hand side of system (16) at $\mu = 0$ on solution (18) yields the matrix

$$A_0(\tau) = \begin{pmatrix} 0 & f_{0\eta}(\eta_0, \tau) \\ \omega'_0(J_{pq}) & 0 \end{pmatrix}.$$

If

$$\omega'_0(J_{pq})f_{0\eta}(\eta_0, \tau) > \sigma_0 > 0, \quad \sigma_0 = \text{const}, \quad \tau \in (-\infty, \infty), \quad (19)$$

then the matrix $A_0(\tau)$ has real eigenvalues of different signs. In this case, as we know [6, chapter 8], in the system

$$\mu \frac{du}{d\tau} = A_0(\tau)u \quad (20)$$

for sufficiently small μ , the space of solutions $U(\mu)$ can be represented in the form

$$U(\mu) = U_+(\mu) + U_-(\mu).$$

For the solutions $u_+(\tau, \mu) \in U_+(\mu)$ the inequality

$$|u_+(\tau, \mu)| \leq M_+ \exp \left[-\frac{\gamma_+}{\mu}(\tau - s) \right] |u_+(s, \mu)|, \quad (-\infty < s < \tau < \infty)$$

holds, and for $u_-(\tau, \mu) \in U_-(\mu)$ the following inequality holds:

$$|u_-(\tau, \mu)| \leq M_- \exp \left[-\frac{\gamma_-}{\mu}(\tau - s) \right] |u_-(s, \mu)|, \quad (-\infty < \tau < s < \infty).$$

Here $M_+, M_-, \gamma_+, \gamma_-$ are positive constants and $|\cdot|$ is a norm in \mathbb{R}^2 . It follows from an estimation of the solutions of system (20) that the solution of this system is unstable for sufficiently small μ if the space $X_-(\mu)$ of the initial conditions of the solutions from $U_-(\mu)$ is non-trivial. Hence, an inhomogeneous system

$$L(\mu) = \mu \frac{dz}{d\tau} - A_0(\tau)z = f(\tau),$$

where $f(\tau)$ is a periodic two-dimensional function, for sufficiently small μ has a unique periodic solution. This solution is represented as

$$z(\tau, \mu) = L^{-1}(\mu)f = \frac{1}{\mu} \int_{-\infty}^{\infty} K(\tau, s, \mu)f(s)ds,$$

where

$$|K(\tau, s, \mu)| \leq M \exp \left(-\frac{\gamma}{\mu}|\tau - s| \right) \quad (-\infty < \tau, s < \infty), \quad (21)$$

and M, γ are positive constants.

We transform system (16) using a change

$$u = \eta - \eta_0(\tau)$$

and write the obtained system in the vector form ($z = (\xi, u)$)

$$\mu \frac{dz}{d\tau} = A_0(\tau)z + H(z, t, \tau, \mu). \quad (22)$$

The components $H(z, t, \tau, \mu)$ have the form

$$\begin{aligned} & f_0(u + \eta_0, \tau) - f_{0\eta}(\eta_0, \tau)u + \mu f_1(u + \eta_0, \tau)\xi + O(\mu^2), \\ & -\mu \frac{d\eta_0}{d\tau} + \frac{1}{2}\mu \omega_{xx}(x_0, \tau)\xi^2 + \mu g_0(u + \eta_0, \tau) + O(\mu^2). \end{aligned}$$

Evidently, the following inequality is valid

$$|H(0, t, \tau, \mu)| \leq p(\mu), \quad (23)$$

where $p(\mu) \rightarrow 0$ as $\mu \rightarrow 0$. The components of the vector function $H(z, t, \tau, \mu)$ are differentiable at z in a sufficiently small neighborhood of the point $(0, 0)$. Therefore the following inequality holds

$$|H(z_1, t, \tau, \mu) - H(z_2, t, \tau, \mu)| \leq p_1(r, \mu)|z_1 - z_2|, \quad (24)$$

where $p_1(r, \mu) \rightarrow 0$ as $r \rightarrow 0$.

The vector function $H(z, t, \varepsilon t, \mu)$ is an almost periodic function of variable t . The problem of almost periodic solutions of system (22) is equivalent to the problem of solvability of the system of integral equations

$$z(\tau, \mu) = \frac{1}{\mu} \int_{-\infty}^{\infty} K(\tau, s, \mu) H(z, s, \mu) ds = \Pi(z, \mu). \quad (25)$$

From inequalities (23), (24) follow that the successive approximations (see, for example, [11])

$$z_0(\tau, \mu) = \Pi(z, 0), \quad z_j(\tau, \mu) = \Pi(z_{j-1}(\tau, \mu), \mu), \quad j = 1, 2, \dots$$

for sufficiently small μ converge uniformly on interval $(-\infty, \infty)$ to a unique almost periodic solution $z^*(\tau, \mu)$ of the system of integral equations (25). The solution $z^*(\tau, \mu)$ tends to $(0, 0)$ uniformly with respect to t as $\mu \rightarrow 0$. Hence, system (22), for sufficiently small μ , has a unique almost periodic solution $z_*(\tau, \mu)$. In its turn, system (16), for sufficiently small μ , has a unique almost periodic solution. Therefore, system (15), for sufficiently small μ , has a unique almost periodic solution. To investigate the stability of the almost periodic solution $z_*(\tau, \mu)$ of system (22), we make a change $z = z_*(\tau, \mu) + y(\tau, \mu)$ and obtain the system

$$\mu \frac{dy}{d\tau} = A_0(\tau)y + H_1(y, t, \tau, \mu), \quad (26)$$

where

$$H_1(y, \psi, \tau, \mu) = H(z_*(\tau, \mu) + y(\tau, \mu), t, \tau, \mu) - H(z_*(\tau, \mu), \psi, \tau, \mu).$$

The problem of the stability of the almost periodic solution $z_*(\tau, \mu)$ is reduced to the problem of the stability of the zero solution of system (26). Taking into account the

exponential estimates on the solutions of system (20), based on the Liapunov's theorem of the instability by first approximation, we obtain that, for sufficiently small μ , the zero solution of system (16) is unstable. Hence, the almost periodic solution $z_*(\tau, \mu)$ of system (22) is unstable. We shall state this result as a theorem applied to system (14).

Let J_{pq} be a constant such that

$$\omega_0(J_{pq}) = 0, \quad \omega'_0(J_{pq}) \neq 0.$$

Let there exists a periodic function $\eta_0(\tau)$ ($0 < \eta_0(\tau) < 2\pi$) such that

$$f_0(\eta_0(\tau), \tau) = 0$$

and inequality (19) holds:

$$\omega'_0(J_{pq})f_{0\eta}(\eta_0, \tau) > \sigma_0 > 0, \quad \sigma_0 = \text{const}, \quad \tau \in (-\infty, \infty).$$

In this case, system (14), for sufficiently small μ , has a unique unstable almost periodic solution.

Hence, given the conditions of Theorem 1 are met in the μ -neighborhood of the resonance point J_{pq} , there exists a unique unstable almost periodic solution.

Now let, instead of inequality (19), the contrary inequality holds

$$\omega_0(J_{pq})f_{0\eta}(\eta_0, \tau) < \sigma_1 < 0, \quad \sigma_1 = \text{const}, \quad -\infty < \tau < \infty. \quad (27)$$

If inequality (27) holds, the eigenvalues of the matrix $A_0(\tau)$ for all τ are purely imaginary. Now we need to consider averaged equations of higher approximations (we used only the first approximation in satisfying inequality (19)).

We consider a narrower neighborhood of the resonance point J_{pq} . We make a change in system (15)

$$\begin{aligned} \xi &= \mu u(t) + \mu \xi_0(\tau) + \mu^3 u_3(t, \tau), \\ \eta &= \eta_0(\tau) + \mu v(t) + \mu^2 v_0(\tau), \end{aligned}$$

where $\eta_0(\tau)$ satisfies equation (17), periodic functions $\xi_0(\tau)$ and $v_0(\tau)$ are solutions of the equations

$$\begin{aligned} \frac{d\eta_0}{d\tau} &= \omega'_0(J_{pq})\xi_0(\tau) + g_0(\eta_0(\tau), \tau), \\ \frac{d\xi_0}{d\tau} &= f_{0\eta}(\eta_0(\tau), \tau)v_0(\tau) + f_1(\eta_0(\tau), \tau)\xi_0(\tau) + \\ &\langle F_1(t, \tau, \eta_0(\tau), J_{pq})u_1(\eta_0(\tau), t, \tau) \rangle_t + \langle F_{0\varphi}(t, \tau, \eta_0(\tau), J_{pq})v_2(\eta_0(\tau), t, \tau) \rangle_t \end{aligned}$$

respectively. Here $\langle \cdot \rangle_t$ is mean value in t for fixed τ . These equations can be solved using the inequalities (19) and (27). The functions $u_3(t, \tau)$ is defined as periodic solution with zero mean value from the equation

$$\begin{aligned} \frac{\partial u_3}{\partial t} &= F_{0\varphi}(t, \tau, \eta_0(\tau), J_{pq})v_2(\eta_0(\tau), t, \tau) + F_1(t, \tau, \eta_0(\tau), J_{pq})u_1(\eta_0(\tau), t, \tau) - \\ &\langle F_{0\varphi}(t, \tau, \eta_0(\tau), J_{pq})v_2(\eta_0(\tau), t, \tau) \rangle_t - \langle F_1(t, \tau, \eta_0(\tau), J_{pq})u_1(\eta_0(\tau), t, \tau) \rangle_t. \end{aligned}$$

The change transforms the system into

$$\begin{aligned}\frac{du}{dt} &= \mu a(\tau)v + \mu^2[b(\tau)u + e(\tau)v^2] + O(\mu^3), \\ \frac{dv}{dt} &= \mu c(\tau)u + \mu^2 d(\tau)v + O(\mu^3),\end{aligned}\tag{28}$$

where

$$\begin{aligned}a(\tau) &= f_{0\eta}(\eta_0(\tau), \tau), \quad b(\tau) = f_1(\eta_0(\tau), \tau), \quad c(\tau) = \frac{1}{2}f_{0\eta\eta}(\eta_0(\tau), \tau), \\ d(\tau) &= \omega'_0(J_{pq}), \quad e(\tau) = g_{0\eta}(\eta_0(\tau), \tau).\end{aligned}\tag{29}$$

Condition (27) in new notation takes the form

$$a(\tau)c(\tau) < \sigma_1 < 0.$$

As was noted, it follows from this condition that the eigenvalues of the matrix of the first approximation are purely imaginary for all τ . We reduce system (28) to "standard form" i.e., to the form where the matrix of the first approximation is zero. A detailed description of reduction of system (28) is contained in [6, chapter 16].

We obtain the system

$$\begin{aligned}\frac{dA}{d\tau} &= \frac{1}{2} \left[b(\tau) + e(\tau) - \frac{h'(\tau)}{h(\tau)} \right] A + f_1(A, B, \chi, \tau) + O(\mu), \\ \frac{dB}{d\tau} &= \frac{1}{2} \left[b(\tau) + e(\tau) - \frac{h'(\tau)}{h(\tau)} \right] B + f_2(A, B, \chi, \tau) + O(\mu),\end{aligned}\tag{30}$$

where

$$h(\tau) = \left[-\frac{d(\tau)}{a(\tau)} \right]^{1/2}, \quad \chi(\tau, \mu) = \frac{1}{\mu} \int_0^\tau [-a(s)c(s)]^{1/2} ds.$$

The functions $f_1(A, B, \chi, \tau)$, $f_2(A, B, \chi, \tau)$ are periodic in χ with period 2π . They contain the terms not lower than quadratic in A and B . Analysis of the system (30) gives the following theorem for the system (14) (see similar Theorem 16.2 in [6]). (Below, the mean value of a periodic function $f(\tau)$ is denoted by $\langle f(\tau) \rangle$.)

Let a resonance point J_{pq} meet the conditions of Theorem 1. Let the inequality

$$a(\tau)d(\tau) < \sigma_1 < 0, \quad t \in (-\infty, \infty).$$

holds. Finally, let the inequality

$$\langle b(\tau) + e(\tau) \rangle \neq 0$$

holds. Then system (14) in the ε -neighborhood of the resonance point, for sufficiently small ε , has a unique almost periodic solution. This solution is asymptotically stable if

$$\langle b(\tau) + e(\tau) \rangle < 0,$$

and unstable if

$$\langle b(\tau) + e(\tau) \rangle > 0.$$

Theorems 1 and 2 apply to the study of resonant solutions in the vibro-impact system (11). The resonance point is defined from equation

$$\omega_0(J_{pq}) = -\frac{\pi\Omega}{\pi - \arctan \frac{J_{pq}}{2\Omega\Delta}} = \frac{q}{p}\nu. \quad (31)$$

From equation (31) follows that

$$\omega'_0(J_{pq}) > 0.$$

To calculate remaining coefficients of the equation (30) we need to specify the function $f(t, \tau)$.

Let the function $f(t, \tau)$ is defined by the formula (8).

To calculate $a(\tau) = f_{0\eta}(\eta_0, \tau)$ we need to average $F_0(t, \tau, \varphi, J_{pq})$. The first term of this function is

$$-4\omega_0(J_{pq})E(\tau)\sin(\nu t + \beta(\tau))\kappa_\psi(\varphi + \frac{q}{p}\nu t). \quad (32)$$

Since

$$\kappa_\psi(\varphi + \frac{q}{p}\nu t) = -\omega_0(J_{pq})^{-1} \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{k \sin k(\varphi + \frac{q}{p}\nu t)}{\Omega_0^2 - k^2}, \quad \Omega_0 = \Omega[\omega_0(J_{pq})]^{-1},$$

we have to average summands in the form

$$E(\tau)\sin(\nu t + \beta(\tau))\sin k(\varphi + \frac{q}{p}\nu t), \quad k = 1, 2, \dots$$

It is easy to see that the mean value of function (32) will be non-zero if and only if $q = 1, p = n$ ($n = 1, 2, \dots$). For $q = 1, p = n$ it equals

$$\frac{2E(\tau)\nu^2}{\pi n(\Omega^2 - \nu^2)} \cos(n\eta - \beta(\tau)).$$

The mean value of the second summand

$$-4\gamma J_{pq}\omega_0^2(J_{pq})\kappa_\psi(\varphi + \frac{q}{p}\nu t)\kappa_\psi(\varphi + \frac{q}{p}\nu t)$$

equals

$$-\frac{\gamma J_{pq}}{2} \left(1 + \frac{4\Omega^2\Delta^2}{J_{pq}^2} \right).$$

In calculating the mean value, we used the following equality

$$\sin^{-2} \pi\Omega_0 = 1 + 4J^{-2}\Omega^2\Delta^2.$$

Consequently, the function $\eta_0(\tau)$ is determined as the solution of the equation

$$\cos(n\eta - \beta(\tau)) = \frac{\gamma J_{pq}\pi n}{4E(\tau)\nu^2}(\Omega^2 - \nu^2) \left(1 + \frac{4\Omega^2\Delta^2}{J_{pq}^2} \right) = A_n(\tau). \quad (33)$$

Since $A_n(\tau) \rightarrow \infty$ as $n \rightarrow \infty$, we see that equation (33) can have solutions only for a finite number of the values of n . If equation (33) has solutions for a given value n ($|A_n(\tau)| < 1$), then these solutions are determined by the following formulas

$$\eta_{0l}(\tau) = \frac{\beta(\tau)}{n} \pm \frac{\arccos A_n(\tau)}{n} + \frac{2l\pi}{n}, \quad l = 0, \dots, n-1.$$

Calculating the derivative of the function $f_0(\eta, \tau)$ at the points $\eta_{0l}(\tau)$ yields

$$f_{0\eta}(\eta_{0l}(\tau), \tau) = \pm \frac{2E(\tau)\nu^2}{\pi(\Omega^2 - \nu^2)} \sqrt{1 - A_n^2(\tau)}, \quad (34)$$

and, therefore, (34) has a positive sign at n points and a negative sign at n points. Straightforward calculations similar to the ones done earlier show that

$$\langle b(\tau) + e(\tau) \rangle = \langle f_1(\eta_0, \tau) + g_{0\eta}(\eta_0, \tau) \rangle < 0.$$

Theorem 1 and 2 imply the following result. If the resonance point J_{1n} is a solution of equation (31), then, if ε is sufficiently small and $|A_n(\tau)| < 1$, equation (11) has n solutions in the $\sqrt{\varepsilon}$ - neighborhood of the resonance point, which are unstable resonant almost periodic in t , and n solutions, which are asymptotically stable resonant almost periodic in t in the ε -neighborhood of the resonance point.

The case when the force is determined by formula (9) is more simple. The functions $f_0(\eta, \tau)$, $f_1(\eta, \tau)$, $g_0(\eta, \tau)$ do not depend on τ . Thus $\eta_0(\tau) \equiv \eta_0 = \text{const}$. We obtain

$$\cos(n\eta - \theta) = \frac{\gamma J_{pq}\pi n}{4A\nu^2}(\Omega^2 - \nu^2) \left(1 + \frac{4\Omega^2\Delta^2}{J_{pq}^2} \right) = A_n$$

and

$$\eta_{0l} = \frac{\theta}{n} \pm \frac{\arccos A_n}{n} + \frac{2l\pi}{n}, \quad l = 0, \dots, n-1.$$

The variable τ is contained only in terms of order $O(\mu^3)$ in systems (15) and (28). We get a result similar to result obtained for the perturbation (8).

6 Conclusion

In this paper we have shown that in the considered vibro-impact system with one degree of freedom under small biharmonic perturbation with close and very different frequencies can occur stable and unstable almost periodic resonant modes.

Our study is based on a combination of the method of averaging on infinite interval [6] and the methods of singular perturbation theory [10,13]. The change of variables of method of averaging transforms the original system into a system with a smooth principal part. This system is then singularly perturbed in the slow time. Further study uses some ideas from the theory of singular perturbations.

Similar methods can be used to study the resonant modes arising from the action of biharmonic perturbation in various problems of theory of oscillation described by equations with smooth and discontinuous coefficients.

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